$$y = \Phi(x) \qquad \Phi : D \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^m$$

$\downarrow$

$$\Phi = \Phi^{(r)} \circ \Phi^{(r-1)} \circ \cdots \circ \Phi^{(1)} \circ \Phi^{(0)}$$

$$\Delta y = D\Phi(x) \cdot \Delta x$$

$$\varepsilon_{y_i} = \sum_{j=1}^{n} \frac{x_j}{\Phi_i(x)} \frac{\partial \Phi_i(x)}{\partial x_j} \varepsilon_{x_j}$$

Condition number

$$\kappa(A) = \|A\| \|A^{-1}\| \geq 1$$

$$y = \Phi(a,b,c) = a+b+c \longrightarrow \frac{x_j}{\Phi_i(x)} \cdot \frac{\partial \Phi_i(x)}{\partial x_j}$$

$$\varepsilon_y = \underbrace{\frac{a}{a+b+c}}\varepsilon_a + \underbrace{\frac{b}{a+b+c}}\varepsilon_b + \underbrace{\frac{c}{a+b+c}}\varepsilon_c$$

amplification factor

$$\varepsilon_{x\pm y} = \frac{x}{x\pm y}\varepsilon_x \pm \frac{y}{x\pm y}\varepsilon_y \qquad \left\{ \left|\frac{x}{x\pm y}\right|, \left|\frac{y}{x\pm y}\right| \right\}$$

$$\varepsilon_{xy} = \varepsilon_x + \varepsilon_y \qquad |\varepsilon_x| \leq eps$$

$$\varepsilon_{x/y} = \varepsilon_x - \varepsilon_y \qquad |\varepsilon_y| \leq eps$$

$$\varepsilon_{\sqrt{x}} = \frac{1}{2}\varepsilon_x$$

$$x \approx -y \Rightarrow \quad x + y \rightarrow$$

از دست دادن ارقام با معنی   Loose of significance

خطای خنثی سازی   Cancellation error

$$y = \phi(p, q) = -p + \sqrt{p^2 + q}$$

$$\frac{\partial \phi}{\partial p} = -1 + \frac{p}{\sqrt{p^2+q}} = \frac{-\sqrt{p^2+q} + p}{\sqrt{p^2+q}} = \frac{-y}{\sqrt{p^2+q}} \rightarrow \frac{p}{y}$$

$$\frac{\partial \phi}{\partial q} = \frac{1}{2\sqrt{p^2+q}} \longrightarrow \frac{q}{y}$$

$$\varepsilon_y = \frac{-p}{\sqrt{p^2+q}}\,\varepsilon_p + \frac{q}{2y\sqrt{p^2+q}}\,\varepsilon_q$$

$$\varepsilon_y = -\frac{p}{\sqrt{p^2+q}}\,\varepsilon_p + \frac{p+\sqrt{p^2+q}}{2\sqrt{p^2+q}}\,\varepsilon_q$$

$$\downarrow$$

$$\left|\frac{-p}{\sqrt{p^2+q}}\right| \leq 1 \,, \qquad \left|\frac{p+\sqrt{p^2+q}}{2\sqrt{p^2+q}}\right| \leq 1$$

$q > 0$ $\rightarrow$ مسأله مطمئن است.

$q \simeq -p^2 \longrightarrow$ مسأله شامل خطای بزرگ خواهد بود.

$p \gg q \Rightarrow$ خطای خنثی سازی اتفاق می‌افتد.

$$x = x^{(0)} \longrightarrow \varphi^{(0)}(x^{(0)}) = x^{(1)} \longrightarrow \cdots \longrightarrow \varphi^{(r)}(x^{(r)}) = x^{(r+1)} = y$$

$\searrow$ elementary operations

$$\varphi^{(i)} = \Phi^{(r)} \circ \cdots \circ \Phi^{(i)}$$

$$\tilde{x}^{(i)} \rightsquigarrow \tilde{x}^{(i+1)} = fl\left(\varphi^{(i)}\left(\tilde{x}^{(i)}\right)\right)$$

$$\Delta x^{(i)} = \tilde{x}^{(i)} - x^{(i)}$$

$$\Delta x^{(i+1)} = fl\left[\varphi^{(i)}\left(\tilde{x}^{(i)}\right)\right] - \varphi^{(i)}\left(x^{(i)}\right)$$

$$= \underbrace{\left\{ fl\left[\varphi^{(i)}\left(\tilde{x}^{(i)}\right)\right] - \varphi^{(i)}\left(\tilde{x}^{(i)}\right) \right\}}_{\alpha_{i+1}} + \underbrace{\left\{ \varphi^{(i)}\left(\tilde{x}^{(i)}\right) - \varphi^{(i)}\left(x^{(i)}\right) \right\}}_{\doteq D\Phi^{(i)}\left(x^{(i)}\right) \cdot \Delta x^{(i)}}$$

$$\phi^{(i)} \quad \text{E.O} \quad \Rightarrow \quad fl(\phi^{(i)}(u)) = rd(\phi^{(i)}(u))$$

$$= \phi^{(i)}(u)(1+\varepsilon)$$

$$x \overset{*}{+} y = rd(x+y)$$

$$|\varepsilon| \leq eps$$

$$\eta_1 = x \overset{*}{+} y = (x+y)(1+\varepsilon_1)$$

$$\eta_1 \overset{*}{+} z = ((x+y)(1+\varepsilon_1)+z)(1+\varepsilon_2) \qquad \neq$$

$$x \overset{*}{+} y \overset{*}{+} z = rd(x+y+z) = (x+y+z)(1+\varepsilon)$$

$$\Phi^{(i)}(u) = \begin{bmatrix} \Phi_1^{(i)} \\ \vdots \\ \Phi_{n_{i+1}}^{(i)} \end{bmatrix}$$

$u \in D_i$, $\Phi^{(i)}(u) \in D_{i+1} \subseteq \mathbb{R}^{n_{i+1}}$

$\Phi_j^{(i)} : D_i \longrightarrow \mathbb{R}$

$$fl(\Phi_j^{(i)}(u)) = rd(\Phi_j^{(i)}(u)) = (1+\varepsilon_j)\Phi_j^{(i)}(u)$$

$|\varepsilon_j| \leq eps$, $j = 1, \dots, n_{i+1}$

$$fl(\Phi^{(i)}(u)) = \begin{bmatrix} fl(\Phi_1^{(i)}(u)) \\ \vdots \\ fl(\Phi_{n_{i+1}}^{(i)}(u)) \end{bmatrix} = \begin{bmatrix} (1+\varepsilon_1)\Phi_1^{(i)}(u) \\ \vdots \\ (1+\varepsilon_{n_{i+1}})\Phi_{n_{i+1}}^{(i)}(u) \end{bmatrix}$$

$$fl\left(\phi^{(i)}(u)\right) = \begin{bmatrix} 1+\varepsilon_1 & & & 0 \\ & 1+\varepsilon_2 & & \\ & & \ddots & \\ 0 & & & 1+\varepsilon_{n_{i+1}} \end{bmatrix} \begin{bmatrix} \phi_1^{(i)}(u) \\ \vdots \\ \phi_{n_{i+1}}^{(i)}(u) \end{bmatrix}$$

$$= \left(I + E_{i+1}\right) \phi^{(i)}(u)$$

I → ماتریس همانی $n_{i+1} \times n_{i+1}$

E_{i+1} →

$$E_{i+1} = \begin{bmatrix} \varepsilon_1 & & & 0 \\ & \varepsilon_2 & & \\ & & \ddots & \\ 0 & & & \varepsilon_{n_{i+1}} \end{bmatrix}$$

$$f(\Phi^{(i)}(u)) - \Phi^{(i)}(u) = E_{i+1} \cdot \Phi^{(i)}(u)$$

$$u \downarrow \tilde{x}^{(i)} \qquad u \downarrow \tilde{x}^{(i)} \qquad u \downarrow \tilde{x}^{(i)}$$

$$f(\Phi^{(i)}(\tilde{x}^{(i)})) - \Phi^{(i)}(\tilde{x}^{(i)}) = E_{i+1} \cdot \Phi^{(i)}(\tilde{x}^{(i)})$$

$$\doteq E_{i+1} \Phi^{(i)}(x^{(i)})$$

$$= E_{i+1} x^{(i+1)}$$

$$= \alpha_{i+1}$$

خطاهایی هستند که جدیداً در مرحله $\Phi^{(i)}$ تولید شده‌اند.

$$\Rightarrow \Delta x^{(i+1)} = \alpha_{i+1} + D\Phi^{(i)}(x^{(i)})\,\Delta x^{(i)} = E_{i+1}\,x^{(i+1)} + D\Phi^{(i)}(x^{(i)})\,\Delta x^{(i)}$$

$i \geq 0$

$$\Delta x^{(0)} = \Delta x$$

$$x = x^{(0)} \longrightarrow x^{(1)} = \Phi^{(0)}(x^{(0)}) \longrightarrow \cdots \longrightarrow y = x^{(r+1)} = \Phi^{(r)}(x^{(r)})$$

$\Delta x^{(0)}$ $\Delta x^{(1)}$ $\Delta x^{(i+1)}$ $\Delta y = \Delta x^{(r+1)}$

$$\Delta x^{(1)} = D\Phi^{(0)}\Delta x + \alpha_1$$

$$\Delta x^{(2)} = D\Phi^{(1)}\Delta x^{(1)} + \alpha_2 = D\Phi^{(1)}\left[D\Phi^{(0)}\Delta x + \alpha_1\right] + \alpha_2$$

$\vdots$

$$\Delta y = \Delta x^{(r+1)} = D\Phi^{(r)}\Delta x^{(r)} + \alpha_{r+1}$$

$$\Delta y = \Delta x^{(r+1)} = \underbrace{D\Phi^{(r)} \cdots D\Phi^{(0)}}_{D\Phi} \Delta x + \underbrace{D\Phi^{(r)} \cdots D\Phi^{(1)}}_{D\psi^{(1)}} \alpha_1 + \cdots + \alpha_{r+1}$$

$$= D\Phi(x)\Delta x + D\psi^{(1)}(x^{(1)})\alpha_1 + \cdots + D\psi^{(r)}(x^{(r)})\alpha_r + \alpha_{r+1}$$

$$= D\Phi(x)\Delta x + \underbrace{D\psi^{(1)}(x^{(1)})E_1 x^{(1)} + \cdots + D\psi^{(r)}(x^{(r)})E_r x^{(r)} + E_{r+1} y}_{\text{total effect of rounding}}$$

که در مقایسه دو الگوریتم، تاثیر کلی خطای گرد کردن را برای هر دو حساب می‌کنیم

و برای هر کدام این خطا کمتر باشد آن را numerically more trustworthy

گوییم.